

SUBGROUP DEPTH AND TWISTED COEFFICIENTS

ALBERTO HERNANDEZ, LARS KADISON AND MARCIN SZAMOTULSKI

ABSTRACT. Danz computes the depth of certain twisted group algebra extensions in [11], which are less than the values of the depths of the corresponding untwisted group algebra extensions in [8]. In this paper, we show that the closely related h-depth of any group crossed product algebra extension is less than or equal to the h-depth of the corresponding (finite rank) group algebra extension. A convenient theoretical underpinning to do so is provided by the entwining structure of a right H -comodule algebra A and a right H -module coalgebra C for a Hopf algebra H . Then $A \otimes C$ is an A -coring, where corings have a notion of depth extending h-depth. This coring is Galois in certain cases where C is the quotient module Q of a coideal subalgebra $R \subseteq H$. We note that this applies for the group crossed product algebra extension, so that the depth of this Galois coring is less than the h-depth of H in G . Along the way, we show that subgroup depth behaves exactly like combinatorial depth with respect to the core of a subgroup, and extend results in [23] to coideal subalgebras of finite dimension.

1. INTRODUCTION AND PRELIMINARIES

Subgroup depth $d_0(H, G)$ of a subgroup H in a finite group G is introduced in [8] as the minimum depth of the induction-restriction table of irreducible characters of H and G , a matrix of nonnegative integers with nonzero rows and columns. As the matrix of induction $K_0(\mathbb{C}H) \rightarrow K_0(\mathbb{C}G)$, the notion of depth also occurs in fields of topological algebra in various guises. In ring theory, the minimum depth $d(B, A)$ of a subring $B \subseteq A$ is introduced in [3] in terms of the natural bimodule ${}_B A_B$ and its tensor powers (and for even depth, tensored one more time by the bimodule ${}_B A_A$ or ${}_A A_B$). The depth $d_0(H, G)$ is recovered in [3] by letting B, A be group algebras over a field of characteristic zero; indeed, [3] shows that depth $d(kH, kG)$ depends only on the characteristic of the field k [3].

1991 *Mathematics Subject Classification.* 16S40, 16T05, 18D10, 19A22, 20C05.

Key words and phrases. entwining structure, Galois coring, coideal subalgebra, subgroup depth, twisted group algebra, core of a subgroup.

The more general definition also allows one to consider depth of integral group ring extensions $\mathbb{Z}H \subseteq \mathbb{Z}G$ and their minimum depth $d_{\mathbb{Z}}(H, G)$. In addition, a combinatorial depth $d_c(H, G)$ is introduced in [3] by using G -set analogues of balanced tensors and bimodules. The following string of inequalities is from [3, 4.5]:

$$d_0(H, G) \leq d_p(H, G) \leq d_{\mathbb{Z}}(H, G) \leq d_c(H, G) \leq 2|G : N_G(H)|.$$

In [21], h-depth of a subring pair $B \subseteq A$ is introduced by the same process as in the definition of depth but focussing on the natural A - A -bimodules of the tensor powers of A relative to B . The minimum h-depth $d_h(B, A)$ is closely related to $d(B, A)$ by the inequality $|d(B, A) - d_h(B, A)| \leq 2$. Its definition in 1.1 suggests the notion of h-depth is natural and almost unavoidable when considering subgroup depth. In this paper, h-depth provides a natural transition (in Section 2) to depth of an A -coring, which is a notion of coalgebra generalized to A -bimodules where the comultiplication and counits are A -bimodule morphisms.

In [11] the subring depth of twisted group algebras of the permutation groups Σ_n are computed as an intriguing contrast to the untwisted case in [8], where it was shown that $d_0(\Sigma_n, \Sigma_{n+1}) = 2n - 1$. In [11] it was shown that with α the nontrivial 2-cocycle (representing the nonzero element in $H^2(\Sigma_n, \mathbb{C}^\times)$), the twisted complex group algebra extensions have minimum depth

$$(1) \quad d(\mathbb{C}_\alpha \Sigma_n, \mathbb{C}_\alpha \Sigma_{n+1}) = 2(n - \lceil \frac{\sqrt{8n+1} - 1}{2} \rceil) + 1.$$

Note that $d(\mathbb{C}_\alpha \Sigma_n, \mathbb{C}_\alpha \Sigma_{n+1}) \leq d_0(\Sigma_n, \Sigma_{n+1})$ with a difference that goes to infinity as $n \rightarrow \infty$. The same is true of the alternating group series A_n , where $d_0(A_n, A_{n+1}) = 2(n - \lceil \sqrt{n} \rceil) + 1 \geq d(\mathbb{C}_\alpha A_n, \mathbb{C}_\alpha A_{n+1})$, the last depth also equal to the right-hand side of Eq. (1); cf. [8, appendix]. Given a subgroup $H \leq G$, we show in this paper that the crossed product of a twisted G -algebra A with subalgebra $A \#_\sigma H$ has h-depth less than or equal to the h-depth of the corresponding group algebra extension: i.e., we establish

$$d_h(A \#_\sigma H, A \#_\sigma G) \leq d_h(kH, kG)$$

in Eq. (32) in Section 4 below. We will also extend an equality for h-depth of a Hopf subalgebra in terms of depth of its quotient module [23] to the equalition for h-depth of a left coideal subalgebra of a Hopf algebra H in Corollary 3.3. Our method is to define depth of corings so that the depth of the Sweedler coring of a ring extension is its h-depth, and apply (Doi-Koppinen) entwining structures that are Galois corings.

In Corollary 4.13 below we show that $d_h(H, G) = d_h(G/N, H/N)$ where N is the core of a subgroup H in a finite group G (over any ground field). In Corollary 1.11 we note that subgroup depth behaves precisely like combinatorial depth with respect to this subgroup N , after proving that subring depth is preserved by quotienting of relatively nice ideals (Theorem 1.7), or better yet, by relatively nice Hopf ideals (Proposition 1.9). A final application is to a left coideal subalgebra R of a finite-dimensional Hopf algebra H , which is left normal iff a nonzero right integral in R is a normal element in H (Theorem 3.6).

1.1. Preliminaries on subalgebra depth. Let A be a unital associative algebra over a field k . The category of modules over A will be denoted by \mathcal{M}_A . (For finite-dimensional A , the notation \mathcal{M}_A denotes the category of finite-dimensional modules.) Two modules M_A and N_A are *similar* (or *H-equivalent*) if $M \oplus * \cong q \cdot N := N \oplus \cdots \oplus N$ (q times) and $N \oplus * \cong r \cdot M$ for some $r, q \in \mathbb{N}$. This is briefly denoted by $M | q \cdot N$ and $N | r \cdot M$ for some $q, r \in \mathbb{N} \Leftrightarrow M \sim N$. Recall that similar modules have Morita equivalent endomorphism rings.

Let B be a subalgebra of A (always supposing $1_B = 1_A$). Consider the natural bimodules ${}_A A_A$, ${}_B A_A$, ${}_A A_B$ and ${}_B A_B$ where the last is a restriction of the preceding, and so forth. Denote the tensor powers of ${}_B A_B$ by $A^{\otimes_B n} = A \otimes_B \cdots \otimes_B A$ (n times A) for $n = 1, 2, \dots$, which is also a natural bimodule over B and A in any one of four ways; set $A^{\otimes_B 0} = B$ which is only a natural B - B -bimodule.

Definition 1.1. Suppose that $A^{\otimes_B(n+1)}$ is similar to $A^{\otimes_B n}$ as natural X - Y -bimodules for subrings X and Y of A and $n \in \mathbb{N}$. One says that subalgebra $B \subseteq A$ has

- **depth** $2n + 1$ if $X = B = Y$ for $n \geq 0$;
- **left depth** $2n$ if $X = B$ and $Y = A$,
- **right depth** $2n$ if $X = A$ and $Y = B$,
- **h-depth** $2n - 1$ if $X = A = Y$,

for $n \geq 1$.

Note that if $B \subseteq A$ has *h-depth* $2n - 1$, the subalgebra has (left or right) *depth* $2n$ by restriction of modules. Similarly, if $B \subseteq A$ has *depth* $2n$, it has *depth* $2n + 1$. If $B \subseteq A$ has *depth* $2n + 1$, it has *depth* $2n + 2$ by tensoring either $-\otimes_B A$ or $A\otimes_B -$ to $A^{\otimes_B(n+1)} \sim A^{\otimes_B n}$. Similarly, if $B \subseteq A$ has *left* or *right depth* $2n$, it has *h-depth* $2n + 1$. Denote the **minimum depth** of $B \subseteq A$ by $d(B, A)$ [3]. Denote the **minimum h-depth** of $B \subseteq A$ by $d_h(B, A)$ [21]. Note that $d(B, A) < \infty \Leftrightarrow d_h(B, A) < \infty$; if so, [21] shows that

$$(2) \quad -2 \leq d(B, A) - d_h(B, A) \leq 1.$$

For example, $B \subseteq A$ has depth 1 iff ${}_B A_B$ and ${}_B B_B$ are similar [5, 22]. In this case, one deduces the following algebra isomorphism,

$$(3) \quad A \cong B \otimes_{Z(B)} A^B,$$

where $Z(B), A^B$ denote the center of B and centralizer of B in A .

Another example is that $B \subset A$ has right depth 2 iff ${}_A A_B$ and ${}_A A \otimes_B A_B$ are similar. If $A = \mathbb{C}G$ is a group algebra of a finite group G and $B = \mathbb{C}H$ is a group algebra of a subgroup H of G , then $B \subseteq A$ has right depth 2 iff H is a normal subgroup of G iff $B \subseteq A$ has left depth 2; a similar statement is true for a Hopf subalgebra $R \subseteq H$ of finite index and over any field [4].

Now let H be a Hopf algebra with counit $\varepsilon : H \rightarrow k$, antipode $S : H \rightarrow H$ and coproduct $\Delta : H \rightarrow H \otimes H, \Delta(h) = h_{(1)} \otimes h_{(2)}$ (Sweedler notation suppressing summations). Let A a *right H -comodule algebra*, i.e., an H -comodule with coaction $\rho : A \rightarrow A \otimes H, \rho(a) = a_{(0)} \otimes a_{(1)}$ that is a unital algebra homomorphism. The coinvariants $\{b \in A \mid \rho(b) = b \otimes 1_H\}$ form a subalgebra B ; one says A is an *H -Galois extension* of B if the Galois mapping $A \otimes_B A \rightarrow A \otimes H$ given by $x \otimes y \mapsto xy_{(0)} \otimes y_{(1)}$ is bijective. Note that the Galois mapping is an isomorphism of natural A - B -bimodules: if $\dim H = n$, then $A \otimes_B A \cong n \cdot {}_A A_B$, and $A \supseteq B$ has depth 2. There is the following remarkable converse growing out of Ocneanu's ideas in subfactor theory with detailed papers by Szymanski, Longo, Nikshych-Vainerman and others:

Theorem 1.2. *Let $A \supseteq B$ be a Frobenius algebra extension with 1-dimensional centralizer and surjective Frobenius homomorphism. If $d(B, A) \leq 2$, then A is a Hopf-Galois extension of B .*

The notion of Frobenius extension is defined in Example 2.3; a short proof-with-references in [20]. The surjectivity condition ensures that A_B is a generator (and conversely [22]). This theorem requires particularly the use of the depth two condition for the construction of a Hopf algebra structure on $\text{End}_B A_B$ or its dual Hopf algebra structure on $\text{End}_A A \otimes_B A_A \cong (A \otimes_B A)^B$ [24]. (However, if $d(B, A) = 1$, none of these difficulties arise, as Eq. (3) forces $A = B$ with one-dimensional center, the trivial Hopf algebra.) The stringent condition on the centralizer A^B may be relaxed if one considers more general Hopf algebras and their Galois coactions (such as weak Hopf algebras and Hopf algebroids) [24].

Note that one always has

$$(4) \quad A^{\otimes_B n} \mid A^{\otimes_B (n+1)}$$

as natural A -bimodules for all $n \geq 2$: one applies the unit mapping and multiplication to obtain a split monic (or split epi). For $n = 1$, though it holds for the other three of the bimodule structures, it is not generally true as A - A -bimodules, $A | A \otimes_B A$ being the separable extension condition on $B \subseteq A$. Now $A \otimes_B A | q \cdot A$ as A - A -bimodules for some $q \in \mathbb{N}$ is the H-separability condition and implies A is a separable extension of B by Hirata, cf. [19, 2.6]. Somewhat similarly, ${}_B A_B | q \cdot {}_B B_B$ implies ${}_B B_B | {}_B A_B$ [22]. It follows that subalgebra depth and h-depth may be equivalently defined by replacing the similarity bimodule conditions for depth and h-depth in Definition 1.1 with the corresponding bimodules on

$$(5) \quad A^{\otimes_B(n+1)} | q \cdot A^{\otimes_B n}$$

for some positive integer q [3, 21, 22].

For example, for the permutation groups $\Sigma_n < \Sigma_{n+1}$ and their corresponding group algebras over any commutative ring K , one has depth $d_K(\Sigma_n, \Sigma_{n+1}) = 2n - 1$ [3]. Depths of subgroups in $PGL(2, q)$, Suzuki groups, twisted group algebra extensions and Young subgroups of Σ_n are computed in [14, 17, 11, 15]. If B and A are semisimple complex algebras, the minimum odd depth is computed from powers of an order r symmetric matrix with nonnegative entries $\mathcal{S} := MM^T$ where M is the inclusion matrix $K_0(B) \rightarrow K_0(A)$ and r is the number of irreducible representations of B in a basic set of $K_0(B)$; the depth is $2n + 1$ if \mathcal{S}^n and \mathcal{S}^{n+1} have an equal number of zero entries [8]. It follows that the subalgebra pair of semisimple complex algebras $B \subseteq A$ always has finite depth.

Similarly, the minimum h-depth of $B \subseteq A$ is computed from powers of an order s symmetric matrix $\mathcal{T} = M^T M$, where s is the rank of $K_0(A)$; the h-depth is $2n + 1$ if \mathcal{T}^n and \mathcal{T}^{n+1} have an equal number of zero entries (equivalently, letting $\mathcal{T}^0 = I_{s \times s}$, one has

$$(6) \quad \mathcal{T}^{n+1} \leq q\mathcal{T}^n$$

for some $q \in \mathbb{N}$) [22, Section 3].

1.2. Depth of Hopf subalgebras, coideal subalgebras and modules in a tensor category \mathcal{M}_H . Let H be a Hopf algebra over an arbitrary field k . Let $R \subseteq H$ be a Hopf subalgebra, so that $\Delta(R) \subseteq R \otimes R$ and the antipode satisfies $S(R) = R$. It was shown in [23, Prop. 3.6] that the tensor powers of H over R , denoted by $H^{\otimes_{R^n}}$, reduce to tensor powers of the generalized quotient $Q := H/R^+H$ as follows:

$$(7) \quad H^{\otimes_{R^n}} \xrightarrow{\cong} H \otimes Q^{\otimes(n-1)}$$

which for $n = 2$ is given by $x \otimes_R y \mapsto xy_{(1)} \otimes \overline{y_{(2)}}$; see [16, Eq. (21)] for the straightforward extension of this to all n . The isomorphism is an H - H -bimodule isomorphism where the left H -module structures are the natural endpoint actions (as well as the right action to the left), and the right H -module structure on $H \otimes Q \otimes \cdots \otimes Q$ is given by the diagonal action of H :

$$h'(y \otimes q_1 \otimes \cdots \otimes q_{n-1}) \cdot h = h'yh_{(1)} \otimes q_1h_{(2)} \otimes \cdots \otimes q_{n-1}h_{(n)}.$$

The following proposition directly makes use of the isomorphism for each $n \geq 2$. Given a subalgebra pair $U \supseteq T$, observe that the bimodule ${}_U U_T$ is projective iff the multiplication mapping $U \otimes T \rightarrow T$, $u \otimes t \mapsto ut$, is a split epi of U - T -bimodules iff there is an element $e = e^1 \otimes e^2 \in U \otimes T$ such that $e^1 e^2 = 1_U$ and $te = et$ for each $t \in T$ (a so-called “right relative separable tower” of algebras $U \supseteq T \supseteq k1_U$).

Proposition 1.3. *Suppose H is a finite-dimensional Hopf algebra with R and Q as above, with intermediate Hopf subalgebras $H \supseteq U \supseteq T \supseteq R$. If ${}_U U_T$ is a projective bimodule, then $d(R, H) < \infty$. In particular, the depth is finite if either H or R is semisimple [22, 23].*

Proof. In a finite tensor category, such as \mathcal{M}_H with the diagonal action, $P \otimes_k X$ is projective if P is projective and X is any H -module [12, Prop. 2.1] (its proof does not need k to be algebraically closed). The proof below will also require the notion of tensor product Hopf algebra $H \otimes K$ of two Hopf algebras H, K , as well as the Hopf opposite algebra (with antipode S^{-1}) [26]. Then Eq. (7) shows (by restriction) that each $H^{\otimes_R n}$ is a projective U - T -bimodule, equivalently projective $H \otimes T^{\text{op}}$ -module, for one extends Q to an (left-sided trivial) U - T -bimodule via $uqt = \varepsilon(u)qt$. Since $R^e = R \otimes R^{\text{op}}$ is a Hopf subalgebra in the finite-dimensional $U \otimes T^{\text{op}}$, and therefore a free extension, it follows that each $H^{\otimes_R n}$ is projective as a natural R - R -bimodule. The Krull-Schmidt Theorem, Eq. (4) and the fact that there are finitely many projective indecomposable isoclasses entail that $H^{\otimes_R(N+1)} \sim H^{\otimes_R(N+2)}$ for N equal to this number (or the number of nonisomorphic simple H - R -bimodules). Thus, $d(R, H) \leq 2N + 3$.

A semisimple Hopf algebra H (or R) is a separable algebra, separability being characterized by the condition ${}_H H$ (or ${}_R R_R$) is projective. The finite depth follows by letting $U = T = H$ (or $U = T = R$). \square

The bimodule isomorphism in Eq. (7) shows quite clearly that the following definition applied to Q will be of interest to computing $d(R, H)$. Let W be a right H -module and $T_n(W) := W \oplus W^{\otimes 2} \oplus \cdots \oplus W^{\otimes n}$.

Definition 1.4. A module W over a Hopf algebra H has **depth** n if $T_{n+1}(W) \mid t \cdot T_n(W)$ for some positive integer t , and depth 0 if W is isomorphic to a direct sum of copies of k_ε , where ε is the counit. Note that this entails that W also has depth $n+1, n+2, \dots$. Let $d(W, \mathcal{M}_H)$ denote its **minimum depth**. If W has a finite depth, it is said to be an algebraic module. If W is an H -module coalgebra, or H -module algebra, the condition of depth n simplifies to $W^{\otimes(n+1)} \mid t \cdot W^{\otimes n}$ for some $t \in \mathbb{Z}_+$ and all $n \in \mathbb{N}$, where $W^{\otimes 0}$ denotes k_ε .

Lemma 1.5. Let I be a Hopf ideal in a Hopf algebra H . Suppose I is contained in the annihilator ideal of an H -module W . Then depth of W is the same over H or H/I .

Proof. The lemma is proven by noting that a Hopf ideal I in $\text{Ann}_H W$ is contained in the annihilator ideal of each tensor power of W , since I is a coideal. Additionally, split epis as in $T_{n+1}(W) \mid t \cdot T_n(W)$ descend and lift along $H \rightarrow H/I$. \square

Recall that the Green ring, or representation ring, of a finite-dimensional Hopf algebra H over a field k , denoted by $A(H)$, is the free abelian group with basis consisting of indecomposable (finite-dimensional) H -module isoclasses, with addition given by direct sum, and the multiplication in its ring structure given by the tensor product. For example, $K_0(H)$ is a finite rank ideal in $A(H)$. As shown in [13], a finite depth H -module W satisfies a polynomial with integer coefficients in $A(H)$, and conversely. Thus, an algebraic module has isoclass an algebraic element in the Green ring, which explains the terminology.

The main theorem in [23, 5.1] proves that Hopf subalgebra (minimum) depth and depth of its generalized quotient Q are closely related by

$$(8) \quad 2d(Q, \mathcal{M}_R) + 1 \leq d(R, H) \leq 2d(Q, \mathcal{M}_R) + 2.$$

Here one restricts Q to an R -module, in order to obtain the better result on depth. If R is a left coideal subalgebra of H , it is not itself a Hopf algebra, and this as a result is unavailable: in this case, we prove below (Corollary 3.3) that the minimum h-depth satisfies

$$(9) \quad d_h(R, H) = 2d(Q, \mathcal{M}_H) + 1.$$

Example 1.6. For the reader who knows something about the Drinfeld double Hopf algebra $D(H)$ of a Hopf algebra H (or group G), we work an example using the notation of $h \in H$ and its dual Hopf subalgebra $f \in H^*$ as subalgebras of $fh \in D(H)$ subject to relations $hf = f(S^{-1}(h_{(3)})h_{(1)})h_{(2)}$ [27, 26]. We compute the quotient Q of

(coopposite) H^* as a Hopf subalgebra of $D(H)$ simply by

$$Q = H^*H/H^{*+}H \cong H$$

via $\overline{f}h \mapsto f(1_H)h$ with right H^* -action $\overline{h}f = f(S^{-1}(h_{(3)})h_{(1)})h_{(2)}$. This gives $\overline{h}f = \overline{h}f(1_H)$ if H is cocommutative.

Thus if H is cocommutative, $d(Q, \mathcal{M}_{H^*}) = 0$, and by the inequality in (8),

$$d(H^*, D(H)) \leq 2,$$

i.e., H^* is a normal Hopf subalgebra in $D(H)$ (recovering some folklore with a hands-off approach). Also by Eq. (9), $d_h(H^*, D(H)) = 3$ since $Q_{D(H)} \cong H_H$ the regular representation, which has depth 1 if $\dim H > 1$ [23]. Minimum depth $d(G, D(G))$ is analyzed in [16].

Finally we remark that if H is semisimple, Eq. (9) in principle computes the depth of the quotient module Q in the finite tensor category \mathcal{M}_H in terms of the symmetric matrix \mathcal{T} in inequality (6) (where $A = H$ and $B = R$ is semisimple [27, ch. 3]): let $d'(\mathcal{T})$ denote the least n for which inequality (6) holds, then

$$(10) \quad d(Q, \mathcal{M}_H) = d'(\mathcal{T})$$

Thus if M denotes the inclusion matrix $K_0(R) \rightarrow K_0(H)$, then $2d'(\mathcal{T}) + 1 = d_{\text{odd}}(M^T)$ (cf. [21, 3.2]) in terms of the minimum (odd) depth of an irredundant nonnegative matrix M (and its transpose M^T) defined in [8, 2.1].

1.3. Depth of a group subalgebra pair compared with its core-free quotient pair. Given a ring A with subring B , say that an A -ideal is *relatively nice* if its intersection, the B -ideal $J = I \cap B$, satisfies $AJ = I$ or $JA = I$. Note that if both set equalities hold, such as when I is a relatively nice Hopf ideal in a finite-dimensional Hopf algebra with respect to a Hopf subalgebra, we are studying a type of normality condition related to the normality notion in [8, Section 4]. Noting the canonical inclusion $B/J \hookrightarrow A/I$, we extend Theorem 3.6 in [25], where I is fully contained in B , to relatively nice ideals.

Theorem 1.7. *Let I be a relatively nice ideal in A , where J is its intersection with B . Minimum depth satisfies $d(B/J, A/I) \leq d(B, A)$.*

Proof. If $d(B, A) = \infty$, there is nothing to prove. Assume that there is a split monomorphism $\sigma : A^{\otimes_B(n+1)} \hookrightarrow q \cdot A^{\otimes_B n}$ of (natural B - or A -) bimodules for some $q, n \in \mathbb{N}$, where we use balanced multilinear notation for the arguments $a_i \in A$ ($i = 1, \dots, n+1$). Let π denote the canonical surjection $A \rightarrow A/I$, where $\pi(a_i) = \overline{a}_i$. Define

$$\overline{\sigma}(\overline{a}_1, \dots, \overline{a}_{n+1}) = q \cdot \pi^{\otimes n} \sigma(a_1, \dots, a_{n+1}) = (\pi^{\otimes n} \sigma_i(a_1, \dots, a_{n+1}))_{i=1}^q.$$

We must show that for any $i = 1, \dots, q$ and $y \in I$,

$$\pi^{\otimes n} \sigma_i(a_1, \dots, y, \dots, a_{n+1}) = 0$$

to see that $\bar{\sigma}$ is well-defined. Suppose without loss of generality that $I = JA$, so there are finitely many $x_{j_{r+1}} \in J$ and $a_{j_{r+1}} \in A$ such that $y = \sum_{j_{r+1}} x_{j_{r+1}} a_{j_{r+1}}$, where we note

$$\sigma_i(a_1, \dots, a_r, y, \dots, a_{n+1}) = \sum_{j_{r+1}} \sigma(a_1, \dots, a_r x_{j_{r+1}}, a_{j_{r+1}}, a_{r+1}, \dots, a_{n+1}),$$

but $a_r x_{j_{r+1}} \in I = JA$. Then this process may be repeated r times, until we have the terms

$$\begin{aligned} \sigma_i(a_1, \dots, y, \dots, a_{n+1}) = \\ \sum_{j_1, \dots, j_{r+1}}^N x_{j_1 \dots j_{r+1}} \sigma(a_{j_1 \dots j_{r+1}}, \dots, a_{j_r j_{r+1}}, a_{j_{r+1}}, a_{r+1}, \dots, a_{n+1}) \end{aligned}$$

where $x_{j_1 \dots j_{r+1}} \in J$, all terms mapping to zero under $\pi^{\otimes n}$. Similarly a splitting for σ descends to $q \cdot (A/I)^{\otimes_{B/J} n} \rightarrow (A/I)^{\otimes_{B/J} (n+1)}$, a splitting for $\bar{\sigma}$. The inequality of minimum depths follows as a consequence of the characterization of depth in (5). \square

Remark 1.8. The proof of Theorem 1.7 may be adapted slightly to show that relative Hochschild cochain groups of a subring pair $B \subseteq A$ with coefficients in an A -bimodule M [18] with left and right annihilator containing a relatively nice ideal I in A satisfies $C^n(A, B; M) \cong C^n(A/I, B/J; M)$ induced by the canonical algebra epi $A \rightarrow A/I$, a cochain isomorphism for all $n \geq 0$.

For the following corollary, assume I is a relatively nice Hopf ideal in a Hopf algebra H .

Proposition 1.9. *The minimum depth satisfies*

$$d(R/J, H/I) \leq d(R, H) \leq d(R/J, H/I) + 1.$$

Moreover, if $d(R/J, H/I)$ is even, the two minimum depths are equal.

Proof. Note that $J = R \cap I$ is a Hopf ideal in R and satisfies both $HJ = I = JH$ by an application of the antipode. Let $Q = H/R^+H$ be the quotient module of $R \subseteq H$, and Q' be the corresponding quotient module of $R/J \subseteq H/I$. Since $J \subseteq R^+$ and so $I \subseteq R^+H$, it follows from a Noether isomorphism theorem that $Q' \cong Q$ as H -modules. Since $(H/R^+H)J = I/R^+H = 0$, it follows from the lemma above that $d(Q, \mathcal{M}_R) = d(Q, \mathcal{M}_{R/J})$. From Eq. (8) it follows that

$$2d(Q, \mathcal{M}_R) + 1 \leq d(R, H), d(R/J, H/I) \leq 2d(Q, \mathcal{M}_R) + 2.$$

The proposition now follows from Theorem 1.7.

If $d(R/J, H/I)$ is even, then $d(R/J, H/I) = 2d(Q, \mathcal{M}_{R/J}) + 2$ from Eq. (8), but the Theorem and Eq. (8) imply that $d(R, H) = d(R/J, H/I)$. \square

Example 1.10. Let H_8 be the 8-dimensional small quantum group (at the root-of-unity $q = i$). This is generated as an algebra by K, E, F such that $K^2 = 1$, $E^2 = 0 = F^2$, $EF = FE$, $FK = -KF$, and $EK = -KE$. Let R_4 be the 4-dimensional Hopf subalgebra generated by K, F , which is isomorphic to the Sweedler algebra. (Please refer to [23, Example 4.9] for the coalgebra structure and details related to depth.) Consider the relatively nice Hopf ideal I with basis $\{F, FK, EF, EFK\}$. Then J is $\text{rad } R$ with basis $\{F, FK\}$. The quotient Hopf algebras $H_8/I \cong R_4$ and $R_4/J \cong \mathbb{C}\mathbb{Z}_2$ have depth $d(\mathbb{C}\mathbb{Z}_2, R_4) = 3$ computed in [34, 4.1]. It follows from the proposition that

$$(11) \quad 3 \leq d(R_4, H_8) \leq 4.$$

Recall that the core $\text{Core}_G(H)$ of a subgroup pair of finite groups $H \leq G$ is the intersection of conjugate subgroups of H ; equivalently, the largest normal subgroup contained in H . Note that if $N = \text{Core}_G(H)$, then $\text{Core}_{G/N}(H/N)$ is the one-element group, i.e., $H/N \leq G/N$ is a *corefree* subgroup.

Corollary 1.11. *Suppose N is a normal subgroup of a finite group G contained in a subgroup $H \leq G$. For any ground field k , ordinary depth satisfies the inequality,*

$$d_k(H/N, G/N) \leq d_k(H, G) \leq d_k(H/N, G/N) + 1.$$

Moreover, if $d_k(H/N, G/N)$ is even, the two minimum depths are equal.

Proof. Note that $I = kGkN^+$ is a relatively nice Hopf ideal in kG (generated by $\{g - gn \mid g \in G, n \in N\}$). (And conversely any Hopf ideal of the group algebra kG is of this form [32].) Also note $kG/I \cong k[G/N]$. The corollary now follows from the previous proposition. \square

This result improves [16, Prop. 3.5] to arbitrary characteristic. Note too that combinatorial depth satisfies a completely analogous property given in [3, Theorem 3.12(d)]. The inequality is in a sense the best result obtainable. For example, let $G = \Sigma_4$, $H = D_8$, the dihedral group of 8 elements embedded in G , and

$$N = \{(1), (12)(34), (13)(24), (14)(23)\}.$$

It is noted after [8, 6.8] that $G/N \cong \Sigma_3$, $H/N \cong \Sigma_2$, and that minimum depth satisfies $d_0(G, H) = 4$ [8, Example 2.6], $d_0(H/N, G/N) = 3$.

2. DEPTH OF CORINGS

Let A be a ring and $(\mathcal{C}, A, \Delta, \varepsilon)$ be an A -coring. Recall that \mathcal{C} is an A -bimodule with coassociative coproduct $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$ and counit $\varepsilon : \mathcal{C} \rightarrow A$, both A -bimodule morphisms satisfying $(\varepsilon \otimes \text{id}_{\mathcal{C}})\Delta = \text{id}_{\mathcal{C}} = (\text{id}_{\mathcal{C}} \otimes \varepsilon)\Delta$ [2]. It follows that $\mathcal{C}^{\otimes_A n} \mid \mathcal{C}^{\otimes_A(n+1)}$ as A -bimodules for each $n \geq 1$. For convenience in notation let $\mathcal{C}^{\otimes_A 0} = A$. The reader may consult [2] for more on corings, their morphisms, their comodules and useful examples, such as coalgebras over ground rings and others we bring up below.

Definition 2.1. An A -coring \mathcal{C} has **depth** $2n+1$ if $\mathcal{C}^{\otimes_A n} \sim \mathcal{C}^{\otimes_A(n+1)}$, where n is a nonnegative integer. If n is a positive integer, \mathcal{C} has depth $2n+1$ if $\mathcal{C}^{\otimes_A(n+1)} \mid m \cdot \mathcal{C}^{\otimes_A n}$ for some $m \in \mathbb{N}$. Note that \mathcal{C} has depth $2n+3$ if it has depth $2n+1$. If \mathcal{C} has a finite depth, let $d(\mathcal{C}, A)$ denote its **minimum depth**.

Example 2.2. Given a ring extension $B \rightarrow A$, let \mathcal{C} denote its Sweedler A -coring $A \otimes_B A$ with coproduct $\Delta : A \otimes_B A \rightarrow A \otimes_B A \otimes_B A (= A^{\otimes_B 3})$ given by $\Delta(a \otimes_B a') = a \otimes 1 \otimes a'$ and counit $\mu : A \otimes_B A \rightarrow A$ by $\mu(a \otimes a') = aa'$ [2]. Note that $\mathcal{C}^{\otimes_A n} \cong A^{\otimes_B(n+1)}$ for integers $n \geq 0$. Therefore, comparing the tensor powers of \mathcal{C} as natural A -bimodules is equivalent to comparing the tensor powers $A^{\otimes_B n}$ as in the definition of h -depth in Definition 1.1. It follows that

$$(12) \quad d(A \otimes_B A, A) = d_h(B, A)$$

An A -coring \mathcal{C} has a *grouplike* element $g \in \mathcal{C}$ if $\Delta(g) = g \otimes_A g$ and $\varepsilon(g) = 1_A$. For example, the Sweedler A -coring $A \otimes_B A$ in the example has a grouplike element $1_A \otimes_B 1_A$. An A -coring \mathcal{C} with grouplike g has invariant subalgebra $A_g^{\text{co}\mathcal{C}} := \{b \in A \mid bg = gb\} := B$ (notation refers to coinvariants of \mathcal{C} -comodule A determined by grouplike [2, p. 278]). Recall that \mathcal{C} is a *Galois coring* if $A \otimes_B A \cong \mathcal{C}$ via $a \otimes_B a' \mapsto aga'$, a coring isomorphism of \mathcal{C} with the Sweedler coring of $B \subseteq A$. For example, the Sweedler coring of a faithfully flat ring extension $A \supseteq B$ is Galois, since $B = \{a \in A \mid a \otimes_B 1_A = 1_A \otimes_B a\}$ follows from A_B or B_A being faithfully flat ([2, 28.6], or see Lemma 3.4).

Example 2.3. Recall that $A \supseteq B$ is a Frobenius extension with $F : A \rightarrow B$ a “Frobenius” homomorphism and $e := \sum_i x_i \otimes_B y_i$ a “dual bases tensor,” satisfying $ae = ea$ for every $a \in A$, if A is a B -coring with coproduct $\Delta : A \rightarrow A \otimes_B A$, $\Delta(a) = ae$ and counit F , this coring being denoted by $\mathcal{C}_{\text{Frob}}$. The more familiar conditions of the counit equations characterize Frobenius extensions [19]. The tensor powers of this coring are now given by natural B -bimodules $\mathcal{C}_{\text{Frob}}^{\otimes_B n} = A^{\otimes_B n}$. Using

Definition 1.1 for the definition of odd depth, we obtain

$$(13) \quad d(\mathcal{C}_{\text{Frob}}, B) = d_{\text{odd}}(B, A)$$

in terms of the minimum odd depth.

3. DEPTH OF COALGEBRA-GALOIS EXTENSIONS

In this section we define depth of a certain coalgebra-Galois extension and see that its minimum depth takes on at least as many interesting values as subgroup depth [4, 3, 8, 11, 14, 15]. In contrast, the minimum depth of a Hopf-Galois extension is one or two.

Let k be a field; all unlabeled tensors are over k . We begin with a review of the entwining structure of an algebra A and coalgebra C . The entwining (linear) mapping $\psi : C \otimes A \rightarrow A \otimes C$ satisfies two commutative pentagons and two triangles (a bow-tie diagram on [2, p. 324]). Equivalently, $(A \otimes C, \text{id}_A \otimes \Delta_C, \text{id}_A \otimes \varepsilon_C)$ is an A -coring with respect to the A -bimodule structure $a(a' \otimes c)a'' = aa'\psi(c \otimes a'')$ (or conversely defining $\psi(c \otimes a) = (1_A \otimes c)a$).

An entwining structure mapping $\psi : C \otimes A \rightarrow A \otimes C$ takes values that may be denoted by $\psi(c \otimes a) = a_\alpha \otimes c^\alpha = a_\beta \otimes c^\beta$, suppressing linear sums of rank one tensors, and satisfies the axioms: (for all $a, b \in A, c \in C$)

- (1) $\psi(c \otimes ab) = a_\alpha b_\beta \otimes c^{\alpha\beta};$
- (2) $\psi(c \otimes 1_A) = 1_A \otimes c;$
- (3) $a_\alpha \otimes \Delta_C(c^\alpha) = a_{\alpha\beta} \otimes c_{(1)}^\beta \otimes c_{(2)}^\alpha$
- (4) $a_\alpha \varepsilon_C(c^\alpha) = a \varepsilon_C(c),$

which is equivalent to two commutative pentagons (for axioms 1 and 3) and two commutative triangles (for axioms 2 and 4), in an exercise.

The following is [2, 32.6] or [9, Theorem 2.8.1].

Proposition 3.1. *Entwining structures $\psi : C \otimes A \rightarrow A \otimes C$ are in one-to-one correspondence with A -coring structures $(A \otimes C, \text{id}_A \otimes \Delta_C, \text{id}_A \otimes \varepsilon_C)$.*

Proof. Given an entwining ψ , the obvious structure maps $(A \otimes C, \text{id}_A \otimes \Delta_C, \text{id}_A \otimes \varepsilon_C)$ form an A -coring with respect to the A -bimodule structure $a(a' \otimes c)a'' = aa'\psi(c \otimes a'')$. Conversely, given an A -coring $A \otimes C$ with coproduct $\text{id}_A \otimes \Delta_C : A \otimes C \rightarrow A \otimes C \otimes C \cong A \otimes C \otimes_A A \otimes C$ and counit $\text{id}_A \otimes \varepsilon_C : A \otimes C \rightarrow A \otimes k \cong A$, one defines $\psi(c \otimes a) = (1_A \otimes c)a$ and checks that ψ is an entwining, and the other details, in an exercise. \square

Our primary example in this section is $A = H$, a Hopf algebra with coproduct Δ , counit ε and antipode $S : H \rightarrow H$, and C a right H -module coalgebra, i.e. a coalgebra $(C, \Delta_C, \varepsilon_C)$ and module C_H satisfying $\Delta_C(ch) = c_{(1)}h_{(1)} \otimes c_{(2)}h_{(2)}$ and $\varepsilon_C(ch) = \varepsilon_C(c)\varepsilon(h)$ for each $c \in C, h \in H$. An entwining mapping $\psi : C \otimes H \rightarrow H \otimes C$ is defined by $\psi(c \otimes h) = h_{(1)} \otimes ch_{(2)}$. The entwining axioms are checked in a more general setup in Section 4.

The associated H -coring $H \otimes C$ has coproduct $\text{id}_H \otimes \Delta_C$ and counit $\text{id}_H \otimes \varepsilon_C$ with H -bimodule structure: $(x, y, h \in H, c \in C)$

$$(14) \quad x(h \otimes c)y = xhy_{(1)} \otimes cy_{(2)}$$

Notice that this is the diagonal action from the right.

Proposition 3.2. *The depth of the H -coring $H \otimes C$ and the depth of the H -module C are related by $d(H \otimes C, H) = 2d(C, \mathcal{M}_H) + 1$.*

Proof. The n -fold tensor product of $H \otimes C$ over H reduces to the H -bimodule isomorphism

$$(15) \quad (H \otimes C)^{\otimes_H n} \cong H \otimes C^{\otimes n}$$

via the mapping

$$\otimes_{i=1}^n h_i \otimes c_i \longmapsto h_1 h_{2(1)} \cdots h_{n(1)} \otimes_{i=1}^{n-1} c_i h_{i+1(i+1)} \cdots h_{n(i+1)} \otimes c_n$$

with inverse $h \otimes c_1 \otimes \cdots \otimes c_n \mapsto h \otimes c_1 \otimes_H 1_H \otimes c_2 \otimes_H \cdots \otimes_H 1_H \otimes c_n$, where $H \otimes C^{\otimes n}$ has right H -module structure from the diagonal action by H : $(h \otimes c_1 \otimes \cdots \otimes c_n)x = hx_{(1)} \otimes c_1 x_{(2)} \otimes \cdots \otimes c_n x_{(n+1)}$. This follows from Eq. (14), cancellations of the type $M \otimes_H H \cong M$ for modules M_H , and an induction on n .

Suppose $d(C, \mathcal{M}_H) = n$, so that $C^{\otimes n} \sim C^{\otimes(n+1)}$ as right H -modules (in the finite tensor category \mathcal{M}_H). Applying an additive functor, it follows that $H \otimes C^{\otimes n} \sim H \otimes C^{\otimes(n+1)}$ as H -bimodules. Applying the isomorphism (15) the coring depth satisfies $d(H \otimes C, H) \leq 2d(C, \mathcal{M}_H) + 1$.

Conversely, if $d(H \otimes C, H) = 2n + 1$, so that $H \otimes C^{\otimes n} \sim H \otimes C^{\otimes(n+1)}$ from Eq. (15) again, we apply that additive functor $k \otimes_H -$ to the similarity and obtain the similarity of right H -modules, $C^{\otimes n} \sim C^{\otimes(n+1)}$. Thus $2d(C, \mathcal{M}_H) + 1 \leq d(H \otimes C, H)$. \square

Now suppose $R \subseteq H$ is a left coideal subalgebra of a finite-dimensional Hopf algebra; i.e., $\Delta(R) \subseteq H \otimes R$. Let R^+ denote the kernel of the counit restricted to R . Then R^+H is a right H -submodule of H and a coideal by a short computation given in [2, 34.2]. Thus $Q := H/R^+H$ is a right H -module coalgebra (with a right H -module coalgebra epimorphism $H \rightarrow Q$ given by $h \mapsto h + R^+H := \bar{h}$). The H -coring $H \otimes Q$ has

grouplike element $1_H \otimes \overline{1_H}$; in fact, [2, 34.2] together with [30] shows that this coring is Galois:

$$(16) \quad H \otimes_R H \xrightarrow{\cong} H \otimes Q$$

via $x \otimes_R y \mapsto xy_{(1)} \otimes \overline{y_{(2)}}$ (also noted in [16] and in [33] for Hopf subalgebras). That H_R is faithfully flat follows from the result that R is a Frobenius algebra and H_R is free [30]. Note that an inverse to (16) is given by $x \otimes \overline{z} \mapsto xS(z_{(1)}) \otimes_R z_{(2)}$ for all $x, z \in H$.

From Proposition 3.2, Eqs. (16) and (12) we note the following.

Corollary 3.3. *For a left coideal subalgebra R in a Hopf algebra H , its h -depth is related to the module depth of Q by*

$$(17) \quad d_h(R, H) = 2d(Q, \mathcal{M}_H) + 1.$$

Proof. The proof is sketched above for finite-dimensional H . For infinite-dimensional H , note that the H -bimodule isomorphism in Eq. (16) remains valid, as does Proposition 3.2 and Eq. (12). \square

Suppose R is a Hopf subalgebra of H . Then Q is an R -module coalgebra by restriction. A similar argument to the one above shows that $d(R, H)$ and $d(Q, \mathcal{M}_R)$ satisfy the inequalities in (8).

Finally we recall that a *C-Galois extension* $A \supseteq B$, where C is a coalgebra and A a right C -comodule via coaction $\delta : A \rightarrow A \otimes C$, the subalgebra of coinvariants is characterized by satisfying $B = \{b \in A \mid \forall a \in A, \delta(ba) = b\delta(a)\}$ and $\beta : A \otimes_B A \mapsto A \otimes C$ given by $\beta(a \otimes a') = a\delta(a')$ is bijective. For example, a left coideal subalgebra $R \subseteq H$ is a coalgebra-Galois extension with respect to the $(H\text{-module})$ coalgebra Q , as sketched above (the details are in [2, 34.2]). Of course, this applies to Hopf subalgebras and more particularly to finite group algebra extensions. Then we see that coalgebra-Galois extensions have at least the range of values computed for subgroup depth [3, 8, 11, 14, 15, 17].

3.1. A faithfully flat interlude. Let C be a coalgebra and H -module quotient of H , with canonical epi $H \rightarrow C$ of right H -module coalgebras, and A is a right H -comodule algebra, with the obvious C -comodule coaction $\delta : A \rightarrow A \otimes C$ given by $\delta(a) = a_{(0)} \otimes \overline{a_{(1)}}$. Define the subalgebra $B = \{b \in A \mid \delta(b) = b \otimes \overline{1_H}\}$. Now suppose that $D \subseteq B$ is a subalgebra for which the canonical (Galois) mapping $\beta : A \otimes_D A \xrightarrow{\cong} A \otimes C$ given by $a \otimes_D a' \mapsto a\delta(a') = aa'_{(0)} \otimes \overline{a'_{(1)}}$, is an isomorphism. If either of the natural modules A_D or ${}_DA$ is faithfully flat, then $D = B$. This follows from noting that for each $b \in B$, $\beta(1_A \otimes_D b) = b \otimes \overline{1_H} = \beta(b \otimes_D 1_A)$ and the next lemma.

Lemma 3.4. *If $b \otimes_D 1 = 1 \otimes_D b$ for $b \in A \supseteq D$ with D a subring of A such that A_D or ${}_D A$ is faithfully flat, then $b \in D$.*

Proof. Recall that a flat module A_D is faithfully flat iff for each module ${}_D N$ such that $A \otimes_D N = 0$, it follows that $N = 0$. Now form the module $N = B'/D$, where $B' = \{b \in A \mid b \otimes_D 1_A = 1_A \otimes b\}$. By an exercise with a commutative square, $A \otimes_D N = 0$, whence $D = B'$. \square

3.2. A normal element characterization of left ad-stability for left coideal subalgebras. Suppose R is left coideal subalgebra of a finite-dimensional Hopf algebra H . Let Q be its quotient right H -module coalgebra defined above. By Skryabin's Freeness Theorem [30] $\dim R$ divides $\dim H$, and (R, ε) is an augmented Frobenius algebra, so R has a nonzero right integral t_R unique up to scalar multiplication [19, Ch. 6]. Then one proves just as in [23, Lemma 3.2] that

$$(18) \quad Q \xrightarrow{\cong} t_R H$$

via $h + R^+ H \mapsto t_R h$.

Recall that a subalgebra R in a Hopf algebra H is stable under the left adjoint action of H if $h_{(1)} r S(h_{(2)}) \in R$ for all $r \in R$, $h \in H$. One briefly says that R is left ad-stable (or left normal [6]). In section 3 of [6], Burciu shows how to prove the following.

Lemma 3.5. *(cf. [6]) A left coideal subalgebra R in a finite-dimensional Hopf algebra H is left ad-stable if and only if $HR^+ \subseteq R^+ H$ if and only if the subalgebra $R \subseteq H$ has right depth 2.*

Proof. For the convenience of the reader, we sketch the proof. If R is left ad-stable in H , $r \in R^+$ and $h \in H$, then $hr = h_{(1)} r S(h_{(2)}) h_{(3)} \in R^+ H$.

If $HR^+ \subseteq R^+ H$, the isomorphism $\beta : H \otimes_R H \xrightarrow{\cong} H \otimes Q$ in Eq. (16) is an H - R -bimodule morphism, since Q_R is trivial. Hence ${}_H H \otimes_R H_R \cong (\dim Q) \cdot {}_H H_R$ and R has right depth 2 in H .

If ${}_H H \otimes_R H \oplus * \cong n \cdot {}_H H_R$, then applying $k \otimes_H -$ yields $Q_R \oplus * \cong n \cdot k_R$. Since $R^+ = \text{Ann } k_R$, it follows easily that $HR^+ \subseteq R^+ H$.

If $HR^+ \subseteq R^+ H$, then $R^+ H$ is a bi-ideal in H , Q is a bialgebra and has a natural H -bimodule structure from multiplication in H . Also β defined above satisfies $\beta(1_H \otimes_R h) = \beta(h \otimes_R 1_H)$ for

$$h \in H^{\text{co}Q} = \{x \in H \mid x_{(1)} \otimes \overline{x_{(2)}} = x \otimes \bar{1}\}.$$

Of course, $R \subseteq H^{\text{co}Q}$. Since β is bijective, $h \otimes_R 1 = 1 \otimes_R h$, thus by Lemma 3.4, $R = H^{\text{co}Q}$. But $H^{\text{co}Q}$ is left ad-stable by an argument used in [27, 3.4.2] modified slightly by the remark in the first sentence of this paragraph. \square

The following generalizes [23, 5.2].

Theorem 3.6. *A left coideal subalgebra R in a finite-dimensional Hopf algebra H is left ad-stable if and only if its right integral t_R is a normal element in H .*

Proof. (\Rightarrow) One argues as in [23, 5.2], using Eq. (18) and Q is under the hypothesis a trivial right R -module, that $h_{(1)}t_RSh_{(2)}$ is a nonzero right integral in R for any $h \in H$, so that

$$ht_R = h_{(1)}t_RS(h_{(2)})h_3 \in t_RH.$$

Hence, $Ht_R \subseteq t_RH$. For the opposite inclusion, note that

$$t_RS(h) = S(h_{(1)})h_{(2)}t_RS(h_{(3)}) \in Ht_R.$$

(\Leftarrow) If $Ht_R = t_RH$, then $\text{Ann } Q_R \supseteq R^+$. Thus $HR^+ \subseteq R^+H$. We conclude that R is left ad-stable in H from Lemma 3.5. \square

One says that a subalgebra R of a Hopf algebra H is right ad-stable if $S(h_{(1)})rh_{(2)} \in R$ for all $h \in H, r \in R$. Applying the antipode anti-automorphism to the above theorem, one obtains (left as an exercise) the correct formulation and proof of the opposite (and equivalent) theorem.

Corollary 3.7. *A right coideal subalgebra is right ad-stable in a finite-dimensional Hopf algebra H iff its left integral t_L is a normal element in H .*

Example 3.8. Consider the 8-dimensional small quantum group $H = H_8$ given as an algebra in Example 1.10, with coalgebra structure given by $\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes 1 + K \otimes E$, and $\Delta(F) = F \otimes K + 1 \otimes F$. It follows that $S(K) = K$, $S(E) = -KE$ and $S(F) = -FK$.

Consider the 2-dimensional Frobenius subalgebra R generated by E (the “ring of dual numbers”). Notice that R is a left coideal subalgebra, since $\Delta(E) \in H \otimes R$. It is left ad-stable since $FEK + ES(F) = 0$. Also $t_R = E$ ($= t_L$) and $HE = EH$ is the 4-dimensional vector subspace of H with basis $\{E, EF, EK, EFK\}$: equivalently, $HR^+ = R^+H$.

However, R is not right ad-stable, since $S(F)EK + EF = 2EF \notin R$. The Frobenius subalgebra of dimension 2 generated by F in H is however a right coideal subalgebra that is right ad-stable with normal integral element, and provides an example of Corollary 3.7. This example does not contradict the correct formulation of the opposite of Lemma 3.5:

Proposition 3.9 (cf. [6]). *A right coideal subalgebra R of H is right ad-stable iff $R^+H \subseteq HR^+$ iff R has left depth 2 in H .*

4. THE CORING OF AN ENTWINING STRUCTURE OF A COMODULE ALGEBRA AND A MODULE COALGEBRA

Let H be a Hopf algebra. Suppose A is a right H -comodule algebra, i.e., there is a coaction $\rho : A \rightarrow A \otimes H$, denoted by $\rho(a) = a_{(0)} \otimes a_{(1)}$ that is an algebra homomorphism and (A, ρ) is a right H -comodule [2, 9, 27]. Moreover, let $(C, \Delta_C, \varepsilon_C)$ be a right H -module coalgebra, i.e., a coalgebra in the tensor category \mathcal{M}_H introduced in more detail in Section 3.

Example 4.1. The Hopf algebra H is right H -comodule algebra over itself, where $\rho = \Delta$. Given a Hopf subalgebra $R \subseteq H$ the quotient module Q defined in Section 1 as $Q = H/R^+H$ is a right H -module coalgebra.

Note that (H, Δ, ε) is also a right H -module coalgebra. The canonical epimorphism $H \rightarrow Q$ denoted by $h \mapsto \bar{h}$ is an epi of right H -module coalgebras, and module Q_H has $\bar{1}_H$ as a cyclic generator.

Of course, if $H = k$ is the trivial one-dimensional Hopf algebra, A may be any k -algebra and C any k -coalgebra.

The mapping $\psi : C \otimes A \rightarrow A \otimes C$ defined by $\psi(c \otimes a) = a_{(0)} \otimes ca_{(1)}$ is an entwining as the reader may easily check (the so-called Doi-Koppinen entwining [2, 33.4], [9, 2.1], which includes the case considered in Section 3).

From Proposition 3.1 it follows that $A \otimes C$ has A -coring structure

$$(19) \quad a(a' \otimes c)a'' = aa'a''_{(0)} \otimes ca''_{(1)}$$

which defines the bimodule ${}_A(A \otimes C)_A$. The coproduct is given by $\text{id}_A \otimes \Delta_C$ and the counit by $\text{id}_A \otimes \varepsilon_C$.

Suppose in this paragraph that H is a finite-dimensional Hopf algebra over an algebraically closed field, in which case \mathcal{M}_H is a finite tensor category [12]. Notice in the equation above that the right A -module is given by a version of the diagonal action in which the category \mathcal{M}_A is a module category over \mathcal{M}_H [1, 12].

Proposition 4.2. *The depth of the A -coring $A \otimes C$ and the depth of the H -module C are related by $d(A \otimes C, A) \leq 2d(C, \mathcal{M}_H) + 1$.*

Proof. The n -fold tensor product of $A \otimes C$ over A reduces to the A -bimodule isomorphism

$$(20) \quad (A \otimes C)^{\otimes_{A^n}} \cong A \otimes C^{\otimes n}$$

via the mapping

$$a \otimes c_1 \otimes \cdots \otimes c_n \mapsto a \otimes c_1 \otimes_A 1_A \otimes c_2 \otimes_A \cdots \otimes_A 1_A \otimes c_n,$$

where $A \otimes C^{\otimes n}$ has right A -module structure from the diagonal action by A : $(a \otimes c_1 \otimes \cdots \otimes c_n)b = ab_{(0)} \otimes c_1b_{(1)} \otimes \cdots \otimes c_nb_{(n)}$. This follows from Eq. (19), cancellations of the type $M \otimes_A A \cong M$ for modules M_A , and an induction on n .

Suppose $d(C, \mathcal{M}_H) = n$, so that $C^{\otimes n} \sim C^{\otimes(n+1)}$ as right H -modules (in the finite tensor category \mathcal{M}_H). Applying an additive functor, it follows that $A \otimes C^{\otimes n} \sim A \otimes C^{\otimes(n+1)}$ as A -bimodules. Then applying the isomorphism (20) obtain $d(A \otimes C, A) \leq 2d(C, \mathcal{M}_H) + 1$. \square

Suppose C has grouplike g' . Then $g = 1_A \otimes g'$ is a grouplike element in $A \otimes C$. Then we have from Eq. (12) and Proposition 4.2 the proof of the following.

Lemma 4.3. *Suppose $A \otimes C := \mathcal{C}$ is a Galois coring with $B = A_g^{coc}$ (or just that B is a subalgebra of A such that $A \otimes_B A \cong \mathcal{C}$ as natural and right-diagonal A -bimodules, respectively). Then the h -depth of the subalgebra $B \subseteq A$ satisfies the inequality, $d_h(B, A) \leq 2d(C, \mathcal{M}_H) + 1$.*

4.1. Crossed products as comodule algebras. Now let A be an associative crossed product $D \#_\sigma H$ for some Hopf algebra algebra H and twisted H -module algebra D with 2-cocycle $\sigma : D \otimes D \rightarrow H$ that is convolution-invertible [27, chapter 7]. Then A is a right H -comodule algebra via $\rho = \text{id}_D \otimes \Delta$, which is quite obvious from the formula for multiplication in $D \#_\sigma H$: ($h, x, y \in H, d, d' \in D$)

$$(21) \quad (d \# h)(d' \# x) = d(h_{(1)} \cdot d')\sigma(h_{(2)}, x_{(1)}) \# h_{(3)}x_{(2)}$$

Additionally, the formulas for D a twisted right H -module and σ a 2-cocycle are useful:

$$(22) \quad h \cdot (x \cdot d) = \sigma(h_{(1)}, x_{(1)})(h_{(2)}x_{(2)} \cdot d)\sigma^{-1}(h_{(3)}, x_{(3)})$$

$$(23) \quad (h_{(1)} \cdot \sigma(x_{(1)}, y_{(1)}))\sigma(h_{(2)}, x_{(2)}y_{(2)}) = \sigma(h_{(1)}, x_{(1)})\sigma(h_{(2)}x_{(2)}, y)$$

Example 4.4. For $H = kG$ a group algebra, one obtains from this the familiar conditions of the group crossed product, $A = D \# kG$, where for $g, h, s \in G$ and $d, d' \in D$,

$$(24) \quad g \cdot (h \cdot d) = \sigma(g, h)(gh \cdot d)\sigma(g, h)^{-1}$$

$$(25) \quad (g \cdot \sigma(h, s))\sigma(g, hs) = \sigma(g, h)\sigma(gh, s)$$

$$(26) \quad (d \# g)(d' \# h) = d(g \cdot d')\sigma(g, h) \# gh$$

Note that $\sigma^{-1}(g, h)$ is interchangeable with $\sigma(g, h)^{-1}$.

Example 4.5. If σ is trivial, $\sigma = 1_D$, then the crossed product reduces to the skew group algebra $D * G$, where D is a G -module and

multiplication is given by $(d\#g)(d'\#h) = d(g \cdot d')\#gh$; this is a well-known setup in Galois theory of fields and commutative algebras. More generally, $D\#_\sigma H$ is clearly isomorphic to the smash product $D\#H$ if $\sigma(x, y) = \varepsilon(x)\varepsilon(y)1_H$: see Eq. (21). Then D is a left H -module algebra. (If the action is Galois, then the endomorphism algebra of D over its invariant subalgebra is isomorphic to the smash product $D\#H$ [27, 8.3.3].)

Example 4.6. If σ is instead nontrivial, but the action of G on D is trivial, i.e. $g \cdot d = d$ for each $g \in G$ and $d \in D$, then $D\#_\sigma kG = D_\sigma[G]$, the twisted group algebra [28], with multiplication given by $(d\#g)(d'\#h) = dd'\sigma(g, h)\#gh$. For example, the real quaternions are a twisted group algebra of \mathbb{R} with $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\sigma = \pm 1$.

Example 4.7. For any group G and normal subgroup N of G , the group algebra is a crossed product of the quotient group algebra acting on the subgroup N as follows [27, 7.1.6]. Let Q denote the group G/N . For each coset $q \in Q$, let $\gamma(q)$ denote a coset representative, choosing $\gamma(\overline{1_G}) = 1_G$. It is an exercise then to show that with $\sigma(\overline{x}, \overline{y}) := \gamma(\overline{x})\gamma(\overline{y})\gamma(\overline{xy})^{-1} \in N$ and action of Q on kN given by $\overline{x} \cdot n = \gamma(\overline{x})n\gamma(\overline{x})^{-1}$, one has $kG = kN\#_\sigma kQ$.

Let $R \subseteq H$ be a Hopf subalgebra pair. Again form the quotient module $Q = H/R^+H$, which is a right H -module coalgebra with canonical epi of right H -module coalgebras $\pi : H \rightarrow Q$, $h \mapsto \overline{h}$. Then the crossed product $A = D\#_\sigma H$, which is a right H -comodule algebra, is also a right Q -comodule via $A \rightarrow A \otimes H \xrightarrow{A \otimes \pi} A \otimes Q$, the composition ρ being the coaction given by $\rho(d\#h) = d\#h_{(1)} \otimes \overline{h_{(2)}}$. Then one sees that $B := D\#_\sigma R \subseteq A^{\text{co}Q}$, the Q -coinvariants defined as the subalgebra $\{b \in A \mid \forall a \in A, \rho(ba) = b\rho(a)\}$; the inclusion follows from noting $r\overline{h} = \varepsilon(r)\overline{h}$ for all $r \in R, h \in H$.

Note that Q has the grouplike element $\overline{1_H}$, so that $g = 1_A \otimes \overline{1_H}$ is a grouplike element in the A -coring $\mathcal{C} = A \otimes Q$. From the previous observation and Eq. (19), it follows that $B = D\#R \subseteq A_g^{\text{co}\mathcal{C}}$.

Suppose that $A \otimes_B A \cong A \otimes Q$ as A -bimodules, or more strongly \mathcal{C} is a Galois A -coring with respect to $B = A_g^{\text{co}\mathcal{C}}$. Then putting together Lemma 4.3 with Corollary 3.3 and Eq. (17), we arrive at the inequality of minimum h-depths,

$$(27) \quad d_h(D\#_\sigma R, D\#_\sigma H) \leq d_h(R, H),$$

the main aim of this paper: we next set about establishing this supposition for finite rank group algebra extensions and certain crossed product Hopf algebra extensions.

Proposition 4.8. *Consider the H -comodule algebra $A = D \#_\sigma H$. Let $B = D \#_\sigma R$ be an R -comodule subalgebra of A , where R is a Hopf subalgebra of H . Suppose the condition (28) below is satisfied. Then $A \otimes_B A \cong A \otimes Q$ as A -bimodules, where $Q = H/R^+H$. If A_B is faithfully flat, then A is a coalgebra Q -Galois extension of B , i.e., $A \otimes Q$ is a Galois A -coring with B the invariant subalgebra.*

Proof. We investigate if the canonical mapping

$$\beta(a \otimes_B a') = aga' = a(d \# h_{(1)}) \otimes \overline{h_{(2)}}$$

(where $a' = d \# h \in D \# H = A$) is bijective, and the module A_B is faithfully flat.

The map $\beta : A \otimes_B A \xrightarrow{\cong} A \otimes Q$ is well-defined: suppose $d' \in D, r \in R$, we check that

$$\beta(a(d' \# r) \otimes_B (d \# h)) = \beta(a \otimes_B (d'(r_{(1)} \cdot d)\sigma(r_{(2)}, h_{(1)}) \# r_{(3)}h_{(2)}))$$

i.e., $a(d' \# r)(d \# h_{(1)}) \otimes \overline{h_{(2)}} = ad'(r_{(1)} \cdot d)\sigma(r_{(2)}, h_{(1)}) \# r_{(3)}h_{(2)} \otimes \overline{r_{(4)}h_{(3)}}$, which is clear since $\overline{r}h = \varepsilon(r)\overline{h}$ for each $r \in R, h \in H$.

We note that $\beta^{-1} : A \otimes Q \rightarrow A \otimes_B A$ is given by

$$\beta^{-1}(a \otimes \overline{h}) = a(\sigma^{-1}(S(h_{(2)}), h_{(3)}) \# S(h_{(1)})) \otimes_B (1_D \# h_{(4)}).$$

The computation $\beta^{-1} \circ \beta = \text{id}_{A \otimes_B A}$ and the computation $\beta \circ \beta^{-1} = \text{id}_{A \otimes Q}$ follow from $\gamma : H \rightarrow A$, defined by $\gamma(h) = 1_D \# h$, having convolution-inverse $\mu : H \rightarrow A$, defined by

$$\mu(h) = \sigma^{-1}(S(h_{(2)}), h_{(3)}) \# S(h_{(1)}),$$

on [27, p. 109]. We are left with verifying that β^{-1} is well-defined, i.e., vanishes when $h = rh'$ for some $r \in R^+, h' \in H$. Dropping the prime, this becomes the condition

$$(28) \quad \sigma^{-1}(S(h_{(2)})S(r_{(2)}), r_{(3)}h_{(3)}) \# S(h_{(1)})S(r_{(1)}) \otimes_B (1_D \# r_{(4)}h_{(4)}) = 0$$

for all $r \in R, h \in H$ such that $\varepsilon(r) = 0$.

Finally, suppose either natural module, A_B or ${}_B A$ is faithfully flat. Then A is a Q -Galois extension of B (cf. [2, 34.10], [29, 1.2, 3.1]). \square

Corollary 4.9. *Let $R \subseteq H$ be a finite-dimensional Hopf subalgebra pair, and A a left H -module algebra. Then h -depth satisfies*

$$(29) \quad d_h(A \# R, A \# H) \leq d_h(R, H).$$

Proof. If $\sigma(x, y) = \varepsilon(x)\varepsilon(y)1_D$ in Proposition 4.8, then the crossed products in the theorem are smash products. The corollary follows if Eq. (28) is satisfied with this choice of σ . But the left-hand side reduces to $1 \# S(h_{(1)}) \otimes_B 1 \# S(r_{(1)})r_{(2)}h_{(2)}$, indeed equal to zero for $r \in R^+$.

Finally, the natural module ${}_RH$ is free by Nichols-Zoeller, so that $A\#H$ is easily shown to be free as a natural left module over $A\#R$. \square

Proposition 4.10. *Suppose G is a group and S is a subgroup of G such that $|G : S| < \infty$. Let $H = kG$, $R = kS$ and A, B be crossed products of the group algebras H and R with left twisted H -module algebra D as above. Then A is a Q -Galois extension of B (where Q is the permutation module of right cosets of S in G).*

Proof. It suffice to check Eq. (28) for $h \in G$ and $r \in S$. Note that $1 - r \in R^+$ and such elements form a k -basis. Also note that

$$\Delta^{n-1}(1 - r) = 1 \otimes \cdots \otimes 1 - r \otimes \cdots \otimes r$$

(n 1's and n r 's on the right-hand side of the equation). Eq. (28) becomes

$$\begin{aligned} \sigma^{-1}(h^{-1}r^{-1}, rh)\#h^{-1}r^{-1} \otimes_B (1_D\#rh) \\ (30) \qquad \qquad \qquad = \sigma^{-1}(h^{-1}, h)\#h^{-1} \otimes_B (1_D\#h) \end{aligned}$$

The inverse of the 2-cocycle Equation (25) is ($\forall x, y, z \in G$)

$$(31) \qquad \sigma^{-1}(xy, z)\sigma^{-1}(x, y) = \sigma^{-1}(x, yz)[x \cdot \sigma^{-1}(y, z)]$$

Letting $x = h^{-1}$, $y = r^{-1}$, $z = rh$, the left-hand side of Eq. (30) becomes

$$\begin{aligned} \sigma^{-1}(h^{-1}, h)[h^{-1} \cdot \sigma^{-1}(r^{-1}, rh)]\sigma(h^{-1}, r^{-1})\#h^{-1}r^{-1} \otimes_B (1_D\#rh) \\ = (\sigma^{-1}(h^{-1}, h)\#h^{-1})(\sigma^{-1}(r^{-1}, rh)\#r^{-1}) \otimes_B (1_D\#rh) \\ = (\sigma^{-1}(h^{-1}, h)\#h^{-1}) \otimes_B (\sigma^{-1}(r^{-1}, rh)\sigma(r^{-1}, rh)\#h) \end{aligned}$$

which equals the right-hand side of Eq. (30).

The natural module ${}_BA$ is free by a short argument using coset representatives g_1, \dots, g_q giving a left R -basis for H . I.e., $|G : S| = q$, $\sum_{i=1}^q Rg_i = H$ and $\sum_{i=1}^q r_i g_i = 0$ implies each $r_i = 0$. Then $\sum_{i=1}^q (D\#_\sigma R)(1_D\#g_i) = \sum_i D\sigma(R, g_i)\#Rh_i = D\#H$, since $\sigma(x, y)$ is invertible in D for all $x, y \in G$. Suppose

$$0 = \sum_{i=1}^q (d_i\#r_i)(1\#g_i) = \sum_i d_i\sigma(r_{i(1)}, g_i)\#r_{i(2)}g_i$$

where each r_i is a linear combination of group elements s_{ij} in S . In this case each $s_{ij}g_i \in Sg_i$ and it follows from the partition of G into left cosets and the equation just above that each $d_i\#r_i = 0$.

Finally $Q = H/R^+H \cong k[S \setminus G]$ by noting that R^+ has k -basis $\{1 - s : s \in S\}$. \square

Theorem 4.11. *Let G be a group with subgroup H of finite index, A a left twisted G -module k -algebra, and $A \#_\sigma G$ an (associative) crossed product. Then h -depth satisfies*

$$(32) \quad d_h(A \#_\sigma H, A \#_\sigma G) \leq d_h(kH, kG).$$

Proof. Follows from the previous proposition and inequality (27). \square

4.2. General results for Hopf algebra H . We may extend Eq. (32) to a finite-dimensional Hopf algebra extension and crossed product algebra extension as follows.

Theorem 4.12. *Suppose H is a finite-dimensional Hopf algebra, with R a Hopf subalgebra, $A = D \#_\sigma H$, $B = D \#_\sigma R$ and $Q = H/R^+H$ as above. Then*

$$(33) \quad d_h(B, A) \leq d_h(R, H).$$

Proof. The proof may be made to follow from [29, 3.6] and Eq. (27), but we provide some more details as a convenience to the reader. Note first that $\beta : A \otimes_B A \rightarrow A \otimes Q$, $x \otimes y \mapsto xy_{(0)} \otimes \overline{y_{(1)}}$ in the proof of Proposition 4.8 is shown to be surjective from the formal inverse β^{-1} defined there and the equation $\beta \circ \beta^{-1} = \text{id}_{A \otimes Q}$.

That β is injective follows from [29] in the following way using norms of augmented Frobenius algebras, freeness of H over R and the augmentation of the convolution algebra Q^* induced from the grouplike element $\overline{1_H} \in Q$. Let B denote $A^{\text{co}Q}$ for this argument. Let λ_H, λ_R be nonzero left integrals on H and R , respectively. Let $\Gamma' \in R$ satisfy $\lambda_R \Gamma' = \varepsilon$, so that Γ' is a nonzero right integral in R . Define $\lambda : Q \rightarrow k$ by $\lambda(\overline{h}) = \lambda_H(\Gamma' h)$, and note the left integral property, $\overline{h_{(1)}} \lambda(\overline{h_{(2)}})$ for all $h \in H$ [29, p. 307, (i)]. By [29, p. 307, (ii)], there is $\Lambda \in H$ such that $\varepsilon_Q = \Lambda \lambda$, which follows from expressing a right norm Γ of λ_H as $\Gamma = h \Gamma'$, then applying a Nakayama automorphism α of H to express $\Lambda = \alpha(h)$. Next define as in [29, p. 303, (1)] $f : {}_B A \rightarrow {}_B B$ by $f(a) = a_{(0)} \lambda(\overline{a_{(1)}})$ using the left integral property.

In order to continue, note that $\text{can} : A \otimes A \rightarrow A \otimes H$, given by $x \otimes y \mapsto xy_{(0)} \otimes y_{(1)}$ is surjective, since given $a \otimes h \in A \otimes H$, $\text{can}(a(\sigma^{-1}(S(h_{(2)}), h_{(3)}) \# S(h_{(1)})) \otimes 1_D \# h_{(4)}) = a \otimes h$ by the computation on [27, p. 109]. It follows from the bijectivity of the antipode that $\text{can}' : A \otimes A \rightarrow A \otimes H$ given by $x \otimes y \mapsto x_{(0)} y \otimes x_{(1)}$ is also surjective; cf. [27, p. 124]. Let $\text{can}'(\sum_i r_i \otimes \ell_i) = 1_A \otimes \Lambda$. Then [29, p. 303, (2)] shows that $a = \sum_i f(ar_i) \ell_i$ for each $a \in A$. Applying this projectivity equation to $\beta(\sum_k x_k \otimes_B y_k) = 0$, [29, p. 303, (3)] shows that $\sum_i x_k \otimes_B y_k = 0$.

Since H is a free R -module, a faithfully flat descent along the crossed product extension shows that $A \#_{\sigma} R = A^{\text{co} Q}$. Since $\beta : A \otimes_B A \xrightarrow{\cong} A \otimes Q$ as A -bimodules, it follows from Eq. (27), and the proof preceding it, that the inequality in the theorem holds. \square

Corollary 4.13. *Suppose N is a normal subgroup of a finite group G contained in a subgroup $H \leq G$. Then the h -depth satisfies the equality, $d_h(kH, kG) = d_h(k[H/N], k[G/N])$.*

Proof. This follows from Corollary 3.3, since in either case $Q \cong k[G \setminus H]$, while N acts trivially on this G -module. Note that $I = kGkN^+$ is a Hopf ideal in kG (generated by $\{g - gn \mid g \in G, n \in N\}$), which annihilates Q , and we apply Lemma 1.5.

A second proof is to apply the observation in Example 4.7 that one has $kG = kN \#_{\sigma} k[G/N]$ and similarly $kH = kN \#_{\sigma} k[H/N]$. Apply now the inequality (27) to obtain $d_h(kH, kG) \leq d_h(k[H/N], k[G/N])$. The inequality $d_h(k[H/N], k[G/N]) \leq d_h(kH, kG)$ has a proof very similar to the proof of Theorem 1.7 using the relatively nice Hopf ideal $I = kGkN^+$. \square

4.3. Acknowledgements. The authors thank Dr. C. Young for stimulating conversations about depth-preserving algebra homomorphisms. Research for this paper was funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT under the project PE-C/MAT/UI0144/2013.nts.

REFERENCES

- [1] N. Andruskiewitsch and J.M. Mombelli, On module categories over finite-dimensional Hopf algebras, *J. Algebra* **314** (2007), 383–418.
- [2] T. Brzezinski and R. Wisbauer, *Corings and Comodules*, Lecture Note Series **309**, L.M.S., Cambridge University Press, 2003.
- [3] R. Boltje, S. Danz and B. Külshammer, On the depth of subgroups and group algebra extensions, *J. Algebra* **335** (2011), 258–281.
- [4] R. Boltje and B. Külshammer, On the depth 2 condition for group algebra and Hopf algebra extensions, *J. Algebra* **323** (2010), 1783–1796.
- [5] R. Boltje and B. Külshammer, Group algebra extensions of depth one, *Algebra Number Theory* **5** (2011), 63–73.
- [6] S. Burciu, Kernels of representations and coideal subalgebras of Hopf algebras, *Glasgow Math. J.* **54** (2012), 107–119.
- [7] S. Burciu and L. Kadison, Subgroups of depth three, *Surv. Diff. Geom.* **XV** (2011), 17–36.
- [8] S. Burciu, L. Kadison and B. Külshammer, On subgroup depth (with an appendix by B. Külshammer and S. Danz), *I.E.J.A.* **9** (2011), 133–166.

- [9] S. Caenepeel, G. Militaru and S. Zhu, Frobenius and separable functors for generalized module categories and nonlinear equations, *Lect. Notes Math.* **1787**, Springer, 2002.
- [10] C.W. Curtis and I. Reiner, *Methods of Representation Theory*, Vol. 1, Wiley Interscience, 1981.
- [11] S. Danz, The depth of some twisted group extensions, *Comm. Alg.* **39** (2011), 1–15.
- [12] P. Etingof and V. Ostrik, Finite tensor categories, *Moscow J. Math.* **4** (2004), 627–654, 782–783.
- [13] W. Feit, *The Representation Theory of Finite Groups*, North-Holland, 1982.
- [14] T. Fritzsche, The depth of subgroups of $\mathrm{PSL}(2, q)$, *J. Algebra* **349** (2011), 217–233. *Ibid* II, *J. Algebra* **381** (2013), 37–53.
- [15] T. Fritzsche, B. Külshammer and C. Reiche, The depth of Young subgroups of symmetric groups, *J. Algebra* **381** (2013), 96–109.
- [16] A. Hernandez, L. Kadison and C. Young, Algebraic quotient modules and subgroup depth, *Abh. Math. Semin. Univ. Hamburg* **84** (2014), 267–283.
- [17] L. Héthelyi, E. Horváth and F. Petényi, The depth of subgroups of Suzuki groups, arXiv preprint 1404.1523.
- [18] G. Hochschild, Relative homological algebra, *Trans. A.M.S.* **82** (1956), 246–269.
- [19] L. Kadison, *New examples of Frobenius extensions*, University Lecture Series **14**, Amer. Math. Soc., Providence, 1999.
- [20] L. Kadison, Depth two and the Galois coring, in: *Noncommutative geometry and representation theory in mathematical physics*, eds. J. Fuchs, A.A. Stolin *et al*, Contemp. Math. **391**, A.M.S., Providence, 2005, 149–156.
- [21] L. Kadison, Odd H-depth and H-separable extensions, *Cen. Eur. J. Math.* **10** (2012), 958–968.
- [22] L. Kadison, Subring depth, Frobenius extensions and towers, *Int. J. Math. & Math. Sci.* **2012**, article 254791.
- [23] L. Kadison, Hopf subalgebras and tensor powers of generalized permutation modules, *J. Pure Appl. Alg.* **218** (2014), 367–380.
- [24] L. Kadison and K. Szlachanyi, Bialgebroid actions on depth two extensions and duality, *Adv. in Math.* **179** (2003), 75–121.
- [25] L. Kadison and C.J. Young, Subalgebra depths within the path algebra of an acyclic quiver, in: *Algebra, Geometry and Mathematical Physics* (AGMP, Mulhouse, France, Oct. 2011), Springer Proc. Math. Stat. **85**, eds. Makhlouf, Stolin *et al*, 2014, 83–97.
- [26] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.
- [27] S. Montgomery, *Hopf Algebras and their Actions on Rings*, C.B.M.S. **82**, A.M.S., 1993.
- [28] D.S. Passman, *The Algebraic Structure of Group Rings*, Dover, 2011, originally publ. 1977, revised 1985.
- [29] H.-J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras, *J. Algebra* **152** (1992), 289–312.

- [30] S. Skryabin, Projectivity and freeness over comodule algebras, *Trans. A.M.S.* **359** (2007), 2597–2623.
- [31] M. Szamotulski, Galois Theory for H-extensions, Ph.D. Thesis, Technical U. Lisbon, 2013.
- [32] D.S. Passman and D. Quinn, Burnside’s theorem for Hopf algebras, *Proc. A.M.S.* **123** (1995), 327–333.
- [33] K.-H. Ulbrich, On modules induced or coinduce from Hopf subalgebras, *Math. Scand.* **67** (1990), 177–182.
- [34] C.J. Young, Depth theory of Hopf algebras and smash products, Uni. Porto Ph.D. Dissertation, 2014.

DEPARTAMENTO DE MATEMATICA, FACULDADE DE CIÊNCIAS DA UNIVERSIDADE DO PORTO, RUA CAMPO ALEGRE 687, 4169-007 PORTO

E-mail address: ahernandeza079@gmail.com, lkadison@fc.up.pt, mszamot@gmail.com